Formation of the post mining subsidence as a process described by stochastic Itô’s Equation

Abstract

The dislocation process of the points located on a surface in the area that had been under the influence of underground mining was observed in a small time scale (short period of time). It exhibited highly irregular behaviour (e.g. the observed points during the time of their vertical subsidence oscillated in a random way). An example of one point is given in the paper to illustrate the nature of the observed irregularities. The observed phenomenon shows that the underlying mechanism which rules the formation process of the post mining subsidence is of the random nature. To describe mathematically the formation process of the post mining subsidence, stochastic Itô’s Equation and, derivable from it, Kolmogorov’s Equation are proposed. The former equation describes the process under discussion from the individual point of view (micro description) and the latter one from the statistical point of view (macro description).

1. Introduction

The extraction of coal from a long wall panel results in major stress changes in the vicinity of the excavation. The produced movements up to ground surface level are three-dimensional, having a vertical and two orthogonal horizontal components. As the position of the surface changes with time, those components also change in value, giving rise to a displacement history of each surface point.

For the practical reasons the period of time between consecutive observations of the surface changes is usually not too small (it varies usually between 3 to 5 months). As a result of that, some property of the surface changes has been overlooked. This was so because in such a long timescale the dynamic of the surface changes reveals only its moderate nature.

It turns out that the surface changes reveal their yet another nature in the case when the timescale is essentially smaller than the above. It shows the results of fifty three surveys of the displacement of surface points which were carried out every eight hours. Both the horizontal and vertical components of the displacement of surface points are highly irregular. We give an example of one point to illustrate the nature of the observed irregularities of the vertical subsidence (see Fig. 2). Clearly these irregularities are biased by the error of the measurement. Nevertheless we are able to prove that a large part of them are of purely random nature. Here we do not present the needed argumentation but only its final result (see Fig. 3).
The observed phenomenon inspired the authors to give its mathematical description with the aid of Ito’s Equation and, derivable from it, Kolmogorov’s Equation. Another source of our inspiration comes from the pioneering paper of J. Litwiniszyn [3].

Finally, we note that we are able to verify the proposed model with empirical data.

2. A sample of results of surveys

The sample of results of surveys concern vertical subsidences of Point No 5 which was located on the surface of an area that had been under the influence of underground mining. Figure 1 shows the location of the point and the face advance of the layer during the time of the quasi-continuous surveys.

The following parameters characterize the underground mining exploitation:
* the duration of the exploitation: 01.06.1998 – 01.07.1999,
* retarded caving: collapse,
* average depth of the excavation: $H_{av} = 370$ [m],
* average thickness of the seam: $g_{av} = 1.4$ [m],
* the duration of the observation of the process: 27.01.1999 – 10.04.1999,
* the duration of the quasi-continuous measurements of the process (by GPS): 15.02.1999 – 04.03.1999.

Fig. 1. Scheme of the development of underground mining exploitation
Rys. 1. Schemat rozwoju podziemnej eksploatacji górniczej
Measurements were carried out with the aid of GPS technology over the period of time from 15.02.1999 to 04.03.1999. Approximately one measurement was performed every eight hours. It follows from the precise evaluation of the error of the used instrument that each result may be biased by an error which does not exceed $\delta_{\text{max}} \leq 6$ [mm].

The results $W_m(t_i)$ of the surveys of the vertical subsidence $W(t)$ of Point No 5 at the instants $t_i (i=1,2,...,53)$ of time are plotted in Figure 2.

One sees that the results are of oscillatory nature. Clearly, the error of the measurement is one source of the oscillations. But the question is whether it is the only one source.

To decide the above problem one calculates the values $W_d(t_i)$, $(i = 1,2,...,53)$, of the deterministic part of the subsidence as follows. We measure with the aid of the precise levelling...
W_{m}(t_1) and W_{m}(t_{53}) (the error of the measurement does not exceed 0.2 [mm]). Then we assume a constant rate of the vertical subsidence.

The calculated values $W_{m}(t_i) - W_{d}(t_i)$ are plotted in Figure 3. Because of the fact that the results of surveys were obtained with the aid of GPS technology, one can assume uniform distribution of the error of measurements.

It follows from this that those of values $W_{m}(t_i) - W_{d}(t_i)$ which satisfy the relation

$$|W_{m}(t_i) - W_{d}(t_i)| > \sigma$$

where:

$$\sigma = \delta_{\text{max}} / \sqrt{3} \approx 3.46$$

stands for the standard deviation ($\delta_{\text{max}} \leq 6$ [mm]), can be counted among those ones whose oscillations are of the genuine stochastic nature.

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Fig. 3. Calculated values of the random part $W_{m}(t_i) - W_{d}(t_i)$ of subsidence, biased by the error of measurement

Rys. 3. Obliczone wartości $W_{m}(t_i) - W_{d}(t_i)$ losowej części obniżeń obciążone błędem pomiaru
3. Probabilistic model

In the absence of any internal perturbation the motion of any point located on the surface of subsidence trough can be described by a differential equation of the form:

\[ dX_t = b(t,X_t) \, dt, \quad X_{t_0} = q, \quad t_0 \leq t < \infty \]  

(3.1)

where:

- \( X_t = (X_{t,1}, X_{t,2}, X_{t,3}) \) stands for the position of the point at time \( t \), \( b(t,x) \) represents the sensitivity to the movement of the rock mass as a whole under the gravitational force at time \( t \) and location \( x \), \( X_{t_0} = q \) is an initial position at the instant \( t_0 \).

In the reality however because of caving a strongly random perturbation is generated by the downward movement of the rock mass. It causes in particular, that any point on the surface of subsidence trough moves in a random way.

Therefore any realistic description of the motion of the point has to take into account the above random perturbation. As an idealized mathematical model of such a random perturbation usually serves Wiener process (also known as a Brownian motion or white noise). The description incorporating the Wiener process element reads as follows:

\[ dX_t = b(t,X_t) \, dt + \sigma(t,X_t) \, dB_t, \quad X_{t_0} = q, \quad t_0 \leq t < \infty \]  

(3.2)

or in the integral form:

\[ X_t = q + \int_{t_0}^{t} b(s,X_s) \, ds + \int_{t_0}^{t} \sigma(s,X_s) \, dB_s, \quad t_0 \leq t < \infty \]  

(3.3)

where:

- \( X_t \) is an \( \mathbb{R}^3 \) valued (unknown) stochastic process defined on \([t_0, T]\), \( b(t,x) \) and \( \sigma(t,x) \) are respectively \( \mathbb{R}^3 \)-valued and \( (3 \times 3 \, \text{matrix}) \)-valued functions, \( B_t \) is a \( 3 \)-dimensional Wiener process over a probability space \((\Omega, \Sigma, \mathcal{P})\) and \( q \) is a random vector independent of \( B_t - B_{t_0} \) for \( t \geq t_0 \).

An equation of the form (3.2) or (3.3) is called an Itô’s stochastic differential equation. The random vector \( q \) is the given initial position at the instant \( t_0 \). Equation (3.2) or (3.3) can be interpreted as the defining equation for an unknown stochastic process \( X_t \) with given initial distribution \( X_{t_0} = q \). The sample paths \( t \to X_t(\omega) \), for \( \omega \in \Omega \), of that process can be regarded as the trajectories of the point on the surface.

The above model enables one to determine the formation of the post mining subsidence trough with the aid of trajectories (histories) of the individual points on the surface. However one has to choose for this purpose very large number of surface points and then to determine their trajectories. Therefore such a procedure is not convenient for the practical reasons. Fortunately, one can pass from the description with the aid of individual trajectories of surface points (micro description) to statistical description (macro description).
The basic notion of the statistical description is transition probability \( P(s, x, t, B) \), \( t_0 \leq s \leq t \leq T \), where \( B \) is a box in \( \mathbb{R}^3 \). It is defined by the relation (see [1, 2]):

\[
P(s, x, t, B) = P(X_t( s, x) \in B)
\]

(3.4)

where:

\( X_t( s, x) \) is the unique Markov process defined by Itô’s equation (3.2) or (3.3) which satisfies the initial condition \( X_s = x \) such that \( P(X_s = x) = 1 \).

Using the above description we are no longer interested in the individual trajectories of surface points. Instead, we want merely to be able to measure that portion of the trajectories which belongs, at the instant \( t \) of time, to a given box \( B \subset \mathbb{R}^3 \), provided the trajectories started earlier, at the instant \( s \leq t \), from a position \( x \in \mathbb{R}^3 \).

In many cases of interest one can assume that the transition probability (3.4) has the form:

\[
P(s, x, t, B) = \int_B p(s, x, t, y) dy
\]

(3.5)

where:

\( p(s, x, t, y) \geq 0 \) is a function such that \( \int_{\mathbb{R}^3} p(s, x, t, y) dy = 1 \).

One can prove that the above function is a fundamental solution of the backward Kolmogorov’s equation (see [1 , 2]). Therefore for fixed \( t \) and \( y \) and for \( s < t \) it satisfies:

\[
\begin{align*}
\frac{\partial}{\partial s} p(s, x, t, y) + Dp(s, x, t, y) &= 0 \\
\text{and} \lim_{s \to t^-} p(s, x, t, y) &= \delta( x - y )
\end{align*}
\]

(3.6)

where

\[
D = \sum_{i=1}^{3} b_i(s, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} b_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j},
\]

\[
\left[ b_{ij}(s, x) \right]_{i,j=1}^{3} = \sigma(s, x) \sigma(s, x)^* \quad \text{and} \quad \left[ b_i(s, x) \right]_{i=1}^{3} = b(s, x).
\]

At the end we note that the above operator \( D \) is connected with Itô’s equation (3.2) or (3.3) by its following coefficients:

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We would like also to note that just the relation (3.6) enables one to calculate the subsidence process (i.e. the macro process).

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Formowanie się górniczej niecki eksploatacyjnej jako proces opisany równaniem Itô

Proces przemieszczeń punktów powierzchni w obszarze oddziaływania podziemnej eksploatacji górniczej był obserwowany w małej skali czasowej (krótki przedział czasu). Proces ten wykazuje wysokie nieregularne zachowanie (na przykład, obserwowane punkty oscylowały w sposób losowy w czasie ich obniżania się). Charakter zaobserwowanych nieregularności został w pracy zilustrowany na przykładzie jednego punktu. Zaobserwowane zjawisko odsłania losowy mechanizm rządzący procesem formowania się niecki pogórniczej. Jako matematyczny opis tego procesu zaproponowano równanie stochastyczne Itô oraz, wyproponowane równanie retrospektywne Kolmogorowa. To pierwsze równanie opisuje powyższy proces z indywidualnego punktu widzenia (micro opis), zaś to drugie równanie opisuje omawiany proces ze statystycznego punktu widzenia (makro opis).

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